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# SPECIAL CURVES IN THREE DIMENSIONAL FINSLER MANIFOLD $F^3$

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Abstract. In this paper, we study AW(k)-type curves in three dimensional Finsler manifold and we give some characterizations related to these curves.

Keywords: AW(k)-type curves, Frenet formulas, Finsler manifold, Bertrand curves.

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## 1. Introduction

Many works have been done related to curves of AW(k)-type [2, 6, 7, 8, 9, 10]. For example, in [10], Özgür and Gezgin studied a Bertrand curve of AW(k)-type in $E^3$ and furthermore, Kulahci, Bektaş and Ergüt investigated curves of AW(k)-type in 3-dimensional null cone and Lorentzian space and they gave some characterizations related to these curves [6-9].

Two curves which, at any point, have a common principal normal vector are called Bertrand curves. The notion of Bertrand curves was discovered by J. Bertrand in 1850. From the past to today a lot of mathematicians have studied Bertrand curves in different areas. Yildirim Yilmaz and Bektaş gave general properties of Bertrand curves in Riemann-Otsuki space [13]. In [5] S. Izumiya and N. Takeuchi considered generic properties of Bertrand curves. Also W. K. Schief investigated the integrability of curves of these types in his paper [11].

In this paper, we have done a study about Bertrand curves of AW(k)-type. However, to the best of author's knowledge, Bertrand curves of AW(k)-type has not been presented in 3-dimensional Finsler manifold. Thus, the study is proposed to serve such a need.

Our paper is organized as follows. In section 2, the basic notions and properties of a Frenet curve are reviewed. In section 3, we study curves of AW(k)-type. We also study bertrand curves of AW(k)-type in section 4.

## 2. Basic notions and properties

Finsler geometry is the most natural generalization of Riemannian geometry. It started in 1918 when P. Finsler wrote his thesis on curves and surfaces in what he called generalized metric spaces. Due to its importance it has a huge research field from geometry to biology, physics and also engineering and computer sciences,[1, 4, 12]. The following part of the study is on the basic concepts of the Finsler manifolds

**Definition 2.1.** *Let $M$ be a real m-dimensional smooth manifold and $TM$ be the tangent bundle of $M$.Denote by $\Pi$ the canonical projection of $TM$ on $M$.*

*Let $M'$ be an non-empty open submanifold of $TM$ such that $\Pi\left(M'\right) = M$ and $\theta\,(M) \cap M' = \varnothing$, where $\theta$ is the zero section of $TM$.*

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*We now consider a smooth function* $F : M' \to (0, \infty)$ *and take* $F^* = F^2$*. Then suppose that for any coordinate system* $\left\{ \left( u', \Phi' \right); x^i, y^i \right\}$ *in* $M'$*, the following conditions are fulfilled:* $(F_1)$ $F$ *is positively homogeneus of degree one with respect to* $\left( y^1, ..., y^m \right)$ *i.e. we have*

$$F\left(x^1, ...x^m, ky^1, ..., ky^m\right) = kF\left(x^1, ..., x^m, y^1, ..., y^m\right)$$

*for any* $(x, y) \in \Phi'\left(U'\right)$ *and any* $k > 0$.

$(F_2)$ *At any point* $(x, y) \in \Phi'\left(U'\right)$

$$g_{ij}(X, Y) = \frac{1}{2}\frac{\partial^2 F^*}{\partial y^i \partial y^j}(X, Y), \; i, j \in \{1, ..., m\}$$

*are the components of a positive definite quadratic form on* $\mathbb{R}^m$, [3].

*We say that the triple* $\mathbb{F}^m = \left( M, \; M', F \right)$ *with satisfying* $(F_1)$ *and* $(F_2)$ *is a Finsler manifold and* $F$ *is the fundamental function of* $\mathbb{F}^m$.

**Definition 2.2.** *Let* $\mathbb{F}^{m+1} = \left( M, M', F \right)$ *be a Finsler manifold and* $\mathbb{F}' = \left( C, C', F \right)$ *be a 1-dimensional Finsler submanifold of* $\mathbb{F}^{m+1}$*, where* $C$ *is a smooth curve in* $M$ *given locally by the equations*

$$x^i = x^i(s) \quad, i \in \{1, ..., m+n\}, \; s \in (a, b)$$

$s$ *being the arclength parameter on* $C$*. Denote by* $(s, v)$ *the coordinates on* $C'$*. Then we have*

$$y^i(s, v) = v\frac{dx^i}{ds} \quad i \in \{0, ..., m\}$$

*Moreover* $\left\{ \frac{\partial}{\partial s}, \frac{\partial}{\partial v} \right\}$ *is a natural field of frames on* $C$ *where* $\frac{\partial}{\partial v}$ *is a unit Finsler vector field,*[3].

**Definition 2.3.** *Let* $\mathbb{F}^3 = \left( M, \; M', F \right)$ *be a 3-dimensional Finsler manifold and* $C$ *be a smooth curve in* $M$ *given locally by the parametric equations*

$$x^i = x^i(s) \;\; ; \;\; \left( x'^1(s), \; x'^2, \; x'^3(s) \right) \neq (0, 0, 0)$$

*where* $s$ *is the arclength parameter on* $C$.

*Then we have*

$$\begin{aligned} &\nabla^*_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v} = kn, \\ &\nabla^*_{\frac{\partial}{\partial s}} n = -k\frac{\partial}{\partial v} + \tau b \\ &\nabla^*_{\frac{\partial}{\partial s}} b = \tau n. \end{aligned} \tag{1}$$

*where* $n$ *and* $b$ *called principal normal Finsler vector field and binormal Finsler vector field on* $C$ *respestively. We are entiteled to call* $\left\{ \frac{\partial}{\partial v}, n, b \right\}$ *be the Frenet frame for the curve* $C$ *in* $\mathbb{F}^3$*. As in the Riemannian case we call* $k$ *the curvature and* $\tau$ *the torsion of* $C$ *respesctively,*[3]

**Proposition 2.1.** *Let* $C$ *be a Frenet curve of* $F^3$ *of osculating order 3 then we have*

$$\nabla^*_{\frac{\partial}{\partial s}} C = \frac{\partial}{\partial v},$$

$$\nabla^*_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v} = kn, \tag{2}$$

$$\nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} C = n.\nabla^*_{\frac{\partial}{\partial s}} k - k^2.\frac{\partial}{\partial v} + k\tau b, \tag{3}$$

$$\nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} C = \left( \nabla^*_{\frac{\partial}{\partial s}} k^2 - k^2 \right) \frac{\partial}{\partial v} +$$
$$+ \left( \nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} k - k^3 + k\tau^2 \right) n +$$
$$+ \left( k\tau + k\nabla^*_{\frac{\partial}{\partial s}} \tau + \tau \nabla^*_{\frac{\partial}{\partial s}} k \right) b.$$

**Notation.** Let us write

$$N_1(s) = kn, \tag{4}$$

$$N_2(s) = n\nabla^*_{\frac{\partial}{\partial s}} k + k\tau b, \tag{5}$$

$$N_3(s) = \left( \nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} k - k^3 + k\tau^2 \right) n + \left( k\tau + k\nabla^*_{\frac{\partial}{\partial s}} \tau + \tau \nabla^*_{\frac{\partial}{\partial s}} k \right) b. \tag{6}$$

**Corollary 2.1.** *$C'(s), C''(s), C'''(s), C''''(s)$ are linearly dependent if and only if $N_1(s)$, $N_2(s)$, $N_3(s)$ are linearly dependent.*

## 3. Curves of AW(k)-type

In this section we consider Frenet curves of AW(k)- type.

**Definition 3.1.** *Frenet curves (of osculating order 3) are* [2]
*i) of type weak AW(2) if they satisfy*

$$N_3(s) = \langle N_3(s), N_2^*(s) \rangle N_2^*(s), \tag{7}$$

*ii) of type weak AW(3) if they satisfy*

$$N_3(s) = \langle N_3(s), N_1^*(s) \rangle N_1^*(s) \tag{8}$$

**Proposition 3.1.** *Let $C$ be a Frenet curve of order 3. If $C$ is of type weak AW(2) then*

$$\nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} k - k^3 - k\tau^2 = 0. \tag{9}$$

**Definition 3.2.** *Frenet curves (of osculating order 3) are* [2]
*i) of type AW(1) if they satisfy $N_3(s) = 0$,*
*ii) of type AW(2) if they satisfy*

$$\|N_2(s)\|^2 N_3(s) = \langle N_3(s), N_2(s) \rangle N_2(s), \tag{10}$$

*iii) of type AW(3) if they satisfy*

$$\|N_1(s)\|^2 N_3(s) = \langle N_3(s), N_1(s) \rangle N_1(s). \tag{11}$$

**Proposition 3.2.** *Let $C$ be a Frenet curve of order 3. Then $C$ is of type AW(1) if and only if*

$$\nabla^*_{\frac{\partial}{\partial s}} \nabla^*_{\frac{\partial}{\partial s}} k - k^3 - k\tau^2 = 0, \tag{12}$$

*and*

$$k\tau + k\nabla^*_{\frac{\partial}{\partial s}}\tau + \tau\nabla^*_{\frac{\partial}{\partial s}}k = 0. \tag{13}$$

**Proposition 3.3.** *Let $C$ be a Frenet curve of order 3. Then $C$ is of type AW(2) if and only if*

$$k\tau\left[\left(\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}}k\right) + \left(\nabla^*_{\frac{\partial}{\partial s}}k\right)\right] + k\left(\nabla^*_{\frac{\partial}{\partial s}}\tau\right)\left(\nabla^*_{\frac{\partial}{\partial s}}k\right) + \tau\left(\nabla^*_{\frac{\partial}{\partial s}}k\right)^2 = k^4\tau + k^2\tau^3. \tag{14}$$

**Proposition 3.4.** *Let $C$ be a Frenet curve of order 3. Then $C$ is of type AW(3) if and only if*

$$k^3\tau + k^3\nabla^*_{\frac{\partial}{\partial s}}\tau + k^2\tau\nabla^*_{\frac{\partial}{\partial s}}k = 0. \tag{15}$$

## 4. Bertrand curves on finsler manifolds

**Definition 4.1.** *Let $C(s)$ and $C^*(s)$ be the regular curves in $F^n$. $N^i_o(s), N^i_1(s), ..., N^i_r(s)$ and $N^{i^*}_o(s), N^{i^*}_1(s), ..., N^{i^*}_r(s)$ are Frenet r-frames respectively on these curves. $C(s)$ and $C^*(s)$ are called Bertrand curves, $N^{i^*}_1(s)$ and $N^i_o(s)$ are linearly dependent. Also $C^*(s)$ is a Bertrand mate for $C(s)$ and $(C, C^*)$ is called a Bertrand couple* [14].

**Definition 4.2.** *A curve $C(s) : I \to F^3$ with $k(s) \neq 0$ is called a Bertrand curve if there exists a curve $C^*(s) : I \to F^3$ such that principal normal lines of $C(s)$ and $C^*(s)$ at $s \in I$ are equal. In this case $C(s)$ is called a Bertrand mate of $C^*(s)$. Any plane curve $C(s)$ is a Bertrand curve whose Bertrand mates are parallel curves of $C(s)$.*

*For a Bertrand curve $C^*(s)$, it has been known that if there exist a point $s_o \in I$ such that $s_o = 0$ then $C^*(s)$ is a plane curve. Therefore the torsion of a spatial Bertrand curve does not vanish,* [5].

**Theorem 4.1.** *Let $(C, C^*)$ be a Bertrand mate in $F^{m+1}$ and $C, C^*$ are given $(I, C)$ and $(I, C^*)$ coordinate neighborhoods respectively. Then* [14]

$$d\left(C(s), C^*(s)\right) = cons\tan t$$

**Theorem 4.2.** *Let $C$ and $C^*$ are given with $(I, C)$ and $(I, C^*)$ coordinate neighborhoods respectively, $C$ and $C^*$ are Bertrand curves if and only if there exist non zero real numbers, such that*

$$\lambda k + \mu\tau = 1$$

*for any $s \in I$,* [14].

**Corollary 4.1.** *Let $C$ be a Frenet curve such that $k(s) \neq 0$, $\tau(s) \neq 0$. Then $C$ is a Bertrand curve if and only if there exist a nonzero real number $\lambda$ such that*

$$\lambda\left(k\tau' - k'\tau\right) - \tau' = 0 \tag{16}$$

**Theorem 4.3.** *Let $C : I \to F^3$ be a Bertrand curve with $k(s) \neq 0$ and $\tau(s) \neq 0$. Then $C$ is of AW(2)-type if and only if there is a non zero real number $\lambda$ such that*

$$\begin{aligned} &k\tau\left[\left(\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}}k\right) + \left(\nabla^*_{\frac{\partial}{\partial s}}k\right) + \lambda k^2\left(\nabla^*_{\frac{\partial}{\partial s}}\tau\right)\left(\nabla^*_{\frac{\partial}{\partial s}}k\right) + \right. \\ &+ \left(\nabla^*_{\frac{\partial}{\partial s}}k\right)^2\tau\left(1 - \lambda k\right) = k^4\tau + k^2\tau^3. \end{aligned} \tag{17}$$

*Proof.* Since $C$ is of AW(2)-type, (14) equality holds and since $C$ is a Bertrand curve, (17) equality holds. If both of these equation are considered, (18) is obtained. □

**Theorem 4.4.** *Let $C : I \to F^3$ be a Bertrand curve with $k(s) \neq 0$ and $\tau(s) \neq 0$. Then $C$ is of AW(3)-type if and only if there is a non zero real number $\lambda$ such that*

$$k^3\tau - \lambda k^4 \nabla^*_{\frac{\partial}{\partial s}} \tau - k^2 \tau \nabla^*_{\frac{\partial}{\partial s}} k \,(1 - \lambda k) = 0. \tag{18}$$

*Proof.* Since $C$ is of AW(3)-type, (16) equality holds and since $C$ is a Bertrand curve, (17) equality holds. If both of these equation are considered, (19) is obtained. □

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